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Micro-Mechanical Approach for Spanwise Periodically and Heterogeneously Beam-like Structures

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Abstract : This paper discusses a refined model for investigating the micro-mechanical behavior of beam-like structures, which are composed of various elastic moduli and complex geometries varying through the cross-section directions and are also periodically-repeated and heterogeneous along the axial direction. Following the previous work (Lee and Yu, 2011), the original three-dimensional static problem is first formulated in a unified and compact form using the concept of decomposition of the rotation tensor. Taking advantage of the smallness of the cross-sectional dimension-to-length parameter and the micro-to-macro heterogeneity, while also performing homogenization along the dimensional reduction simultaneously, the variational asymptotic method is rigorously used to construct a total energy function, which is asymptotically correct up to the second order. Furthermore, through the transformation procedure based on the pure kinematic relations and the linearized equilibrium equations, a generalized Timoshenko model is systematically established. For the purpose of dealing with realistic and complex geometries and constituent materials at the microscopic level, this present approach is incorporated into a commercial analysis package. A few examples available in literature are used to demonstrate the consistency and efficiency of this proposed model, especially for the structures, in which the effects of transverse shear deformations are significant.

Key Words : Homogenization, Dimensional reduction, Variational asymptotic method, Generalized Timoshenko model, Beam-like structures, Transverse shear deformations

1. Introduction

With the help of the phenomenal power of present day computer facilities, the full three-dimensional (3D) finite element analysis (FEA) is widely accepted and used for the analysis of new microstructure-based composite structures. However, it is not an efficient and convenient method because of the inordinate requirements of computational costs which are needed to capture the micro-scale mechanical characteristics by meshing all of the details of constituent microstructures.

For this reason, research attentions devoted

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to an alternative approach of the full 3D FEA, especially, by using a unit cell (UC) have received considerable attention in the past several decades.[1] However, since most of those proposed approaches consider the composite structure as a periodic assembly of many UCs along the whole direction, those are not suitable for use to analyze commonly used engineering structures, such as dimensionally reducible structures.[2,3] In recent years, the formal asymptotic method[2] and the computational homogenization method[3] have been applied in order to study this problem. Although both are mathematically elegant and rigorous in developing a simple engineering model (e.g. the Euler-Bernoulli beam model), it is not easy to extend this approach to yield a refined model (e.g. the Timoshenko beam model) without any kinematic and boundary condition assumptions. Therefore, those two methods are still limited to the simple engineering model under the assumption of the Euler-Bernoulli beam model.

In this paper, the previous work[4] is extended by producing the refined model. The 3D anisotropic elasticity problem is first formulated in an intrinsic form using the concept of decomposition of the rotation tensor.[5] Considering the smallness of the cross-sectional dimension-to-length (e) and heterogeneity (η), the variational asymptotic method (VAM)[6] is then used to rigorously decouple the original 3D anisotropic, heterogeneous problem into (i) a linear 3D micro-mechanical analysis, and (ii) a nonlinear one-dimensional (1D) macro-beam analysis. Unlike the Euler-Bernoulli beam model introduced into the previous work, the new micro-mechanical model is based on an asymptotically correct energy function up to the second order, and then transformed into the generalized Timoshenko beam model which is capable of capturing the transverse shear deformations. To use in the numerical procedure, the present approach can be implemented in COMSOL MULTIPHYSICS™ (COMSOL), a finite element based simulation and modeling tool. Several examples, in which the transverse shear deformation is especially significant, are used to demonstrate the application and accuracy of this new model.

2. 3D beam kinematics and 3D energy formulation under homogenization

Let us consider a heterogeneous beam-like structure composed of periodically-repeated unit cells (UCs), denoted by Ωs, over the axial coordinate x_1 along a 1D reference line r (see Fig. 1). Here and throughout the paper, Greek indices assume values 2 and 3 while Latin indices assume 1, 2, and 3.

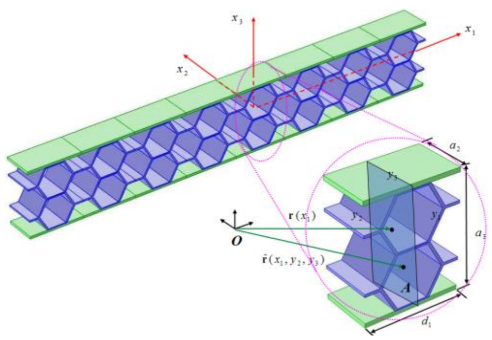

Fig. 1 A heterogeneous beam-like structure with spanwise-repeated unit cell

For the 3D beam kinematic description, one can represent the position of any material point by its position vector $\hat{\mathbf{r}}$ relative to a point O fixed in an inertial frame, such that

$$\hat{\mathbf{r}}(x_1;y_\alpha)=\mathbf{r}(x_1)+\eta y_\alpha \mathbf{b}_\alpha(x_1) \quad \text{with } x_i=\eta y_i \tag{1}$$

where $\mathbf{r}$ is the position vector of a point located by x_1 on the reference line, and $\mathbf{b}_i$ denotes a local orthonormal reference triad along the macroscopic coordinate lines in the undeformed configuration.

When the beam-like structure is deformed, the particle that had position vector $\hat{\mathbf{r}}$ in the undeformed state now has the following position vector $\hat{\mathbf{R}}$ in the deformed configuration with a new orthonormal triad $\mathbf{T}_i$

$$\hat{\mathbf{R}}(x_1;y_i)=\mathbf{R}(x_1)+\eta y_\alpha \mathbf{T}_\alpha(x_1)+\eta w_i(x_1;y_i)\mathbf{T}_i(x_1) \tag{2}$$

where $\mathbf{R}=\mathbf{r}+\mathbf{u}$ denotes the position vector for the deformed structure, $\mathbf{u}=u_i\mathbf{b}_i$ is the displacement vector of the reference line, and w_i denotes the undetermined fluctuating functions describing any deformations not captured by $\hat{\mathbf{R}}$ and $\mathbf{T}_i$.

In order to ensure a one-to-one mapping between $\hat{\mathbf{R}}$ and ($\mathbf{R}$, $\mathbf{T}_i$, and w_i) in Eq.(2), five no rigid-body constraints and y_1-periodic condition of w_i are required

$$\langle w_i\rangle=0 \text{ and } \langle w_{2|3}-w_{3|2}\rangle=0$$
$$w_i(x_1;d_1/2,y_\alpha)=w_i(x_1;-d_1/2,y_\alpha) \tag{4}$$

For the purpose of formulating the problem in an intrinsic form at the macroscopic level, the 1D classical strain measures can be defined as follows

$$\mathbf{R}'=(1+\gamma_{11})\mathbf{T}_1 \text{ and } \mathbf{T}_1{}'=\mathbf{K}\times\mathbf{T}_i \quad \text{with } (\ )'=\partial(\ )/\partial x_1 \tag{5}$$

where γ_{11} is the extensional strain and $\mathbf{K}$ (κ_i) is the curvature vector of the deformed reference line.

Based on the concept of decomposition of the rotation tensor,[5] the 3D Jauman-Biot-Cauchy strain components for small local rotation are represented in the following form

$$\Gamma_{ij}=\tfrac{1}{2}(F_{ij}+F_{ji})-\delta_{ij} \tag{6}$$

where δ_{ij} denotes the Kronecker symbol and F_{ij} the deformation gradient tensor.

Under the essential assumption that the 1D classical strains are small compared to unity, the 3D strain field can be represented as

$$\begin{aligned}\Gamma_{11}&=\gamma_{11}+\eta_{\beta\alpha}y_\alpha\kappa_\beta+\eta w_1'+w_{1|1}\\ 2\Gamma_{1\alpha}&=\eta_{\beta\alpha}y_\beta\kappa_1+\eta w_\alpha'+w_{1|\alpha}+w_{\alpha|1}\\ \Gamma_{\alpha\beta}&=w_{\alpha|\beta}\end{aligned} \tag{7}$$

with $(\ )_{|i} = \partial(\ )/\partial y_i$, $\eta_{22} = \eta_{33} = 0$ and $\eta_{23} = -\eta_{32} = 1$

The strain energy stored in the heterogeneous beam-like structure can be obtained as:

$$U = \int_0^l \mathcal{U} dx_1 \text{ with } 2\omega \mathcal{U} = \left\langle \Gamma^T D \Gamma \right\rangle \quad (8)$$

with $\Gamma = \lfloor \Gamma_{11}\ 2\Gamma_{12}\ 2\Gamma_{13}\ \Gamma_{22}\ 2\Gamma_{23}\ \Gamma_{33} \rfloor^T$

where $\mathcal{U}$ is the strain energy per unit span, ω is the length of the UC in the y_1 direction, and D is the 3D material matrix, which consists of elements of the fourth-order elasticity tensor expressed in the microscopic coordinate system y_i.

To deal with the applied loads, we first leave the existence of a potential energy open and then alternatively define the 3D virtual work of the structure, such as[7)]

$$\overline{\delta W} = \overline{\delta W}_{1D} + \eta \overline{\delta W}^*_e + \eta^2 \overline{\delta W}^*_s \quad (9)$$

where $\overline{\delta W}_{1D}$ is the virtual work not related with the fluctuating functions, while $\overline{\delta W}^*_e$ and $\overline{\delta W}^*_s$ are the virtual works related with ones in $\mathbf{T}_1$ and $\mathbf{T}_\alpha$, respectively. See Ref. 4 for the entire expressions in detail of Eq.(9).

Now, the complete statement of the problem can be presented in terms of the principle of virtual work, and represented as follows

$$\delta\Pi = 0 \text{ with } \Pi = U - (\eta W^*_e + \eta^2 W^*_s) \quad (10)$$

As shown in many literatures,[4,7)] VAM can be used to calculate the 3D unknown fluctuating functions asymptotically without significant loss of accuracy or significant burn of computational costs.

3. Dimensional reduction

To rigorously and efficiently reduce the original 3D problem to the effective 1D beam one, VAM will be first used to reproduce the 3D potential energy stored in the heterogeneous beam-like structure into the intrinsic formulation in terms of 1D classical strain measures. Then it will estimate asymptotically correct solutions up to the desired order taking advantage of the small parameters inherent in the structure.[4)]

3.1 Zeroth-order approximation

With the asymptotic order assessment introduced into the previous work,[4)] the virtual work done by the external forces in Eq. (9) can be negligible in the zeroth-order approximation because the applied loads are of a higher order. Thus, the total potential functional in Eq. (10) can be simply expressed as:

$$\delta\Pi = 0 \text{ with } \delta U = 0 \text{ or } \delta\hat{\Pi} \text{ with } \delta\mathcal{U} = 0 \quad (11)$$

In order to perform the numerical procedure, the unknown fluctuating functions can be alternatively represented by $w(x_1; y_i) = V_0(y_i)\varepsilon(x_1)$, with $w = \lfloor w_1\ w_2\ w_3 \rfloor^T$ and $\varepsilon = \lfloor \gamma_{11}\ \kappa_1\ \kappa_2\ \kappa_3 \rfloor^T$. We can then calculate the energy functional storing in the UC, which is asymptotically correct through the order of $\mu\hat{\varepsilon}^2$ as

$$2\omega\hat{\Pi}^0_\Omega = \varepsilon^T \langle [\Gamma_h V_0]^T D[\Gamma_e] + [\Gamma_e]^T D[\Gamma_e] \rangle \varepsilon = \varepsilon^T \bar{A}\varepsilon \quad (12)$$

where $\bar{A}$ is the 4×4 effective beam stiffness calculated from knowledge of complex geometric and material characteristics in a representative UC at the microscopic level considering the smallness of both the cross-section dimension-to-length parameter and the heterogeneity. Note that the present approach extends the previous work in the same framework, see Refs. 4 and 7 for a detailed derivation of how to calculate the fluctuating functions up to the second order.

3.2 First-order approximation

To obtain the higher-order approximation, we simply perturb the zeroth-order outcome, resulting in the following form

$$w = w_0 + \eta w_1 \text{ with } w_0 = V_0\varepsilon \quad (13)$$

In a manner similar to the procedure to obtain the total potential energy functional on the zeroth-order approximation, one can calculate the following, which is asymptotically correct by up to one to the order of $\mu\eta^2(h/l)^2\hat{\varepsilon}^2$ as

$$2\omega\hat{\Pi}^1_\Omega = 2\varepsilon^T\bar{A}\varepsilon + 2\varepsilon^T\bar{B}\varepsilon' + \varepsilon'^T\bar{C}\varepsilon' + 2\varepsilon^T\bar{D}\varepsilon'' - 2\varepsilon^T\bar{F}_\varepsilon - 2\varepsilon'^T\bar{F}_{\varepsilon'} \quad (14)$$

Note that these stiffness matrices are not explicitly presented since the numerical expressions are quite lengthy.[4,7]

3.3 Transformation to Generalized Timoshenko Model

On the mathematical perspective, the energy formulation in Eq. (14) is asymptotically correct up to the second order. However, it is not a practical form for the engineering application due to the appearance of the classical strain measure derivatives which result in more complicated than necessary boundary conditions. Therefore, we need to transform it into a commonly used engineering model such as the following Timoshenko model

$$2\omega\hat{\Pi}_\Omega = \varepsilon_t^T\bar{X}\varepsilon_t + 2\varepsilon_t^T\bar{Y}\gamma_s + \gamma_s^T\bar{G}\gamma_s - 2\varepsilon_t^T\bar{F}_{\varepsilon_t} - 2\gamma_s^T\bar{F}_{\gamma_s} \quad (15)$$

with $\varepsilon_t = \lfloor \gamma^*_{11}\ \kappa^*_1\ \kappa^*_2\ \kappa^*_3 \rfloor^T$ and $\gamma_s = \lfloor 2\gamma^*_{12}\ 2\gamma^*_{13} \rfloor^T$

Following the kinematic relationship between the classical strain measures, ε, and the refined strain measures, ε_t and γ_s, and the equilibrium equation-based expressions for

ε, ε', and ε'' in terms of ε_t and γ_s, we can construct the generalized Timoshenko's stiffness matrices ($\overline{X}$, $\overline{Y}$, and $\overline{G}$). See Ref. 7 for a detailed derivation to obtain these stiffness matrices.

4. Validation examples

As with validation examples studied in Dai and Zhang[8], three sandwich beams with periodically variable cross-sections are used to demonstrate the proposed theory's application and accuracy in Fig. 2:

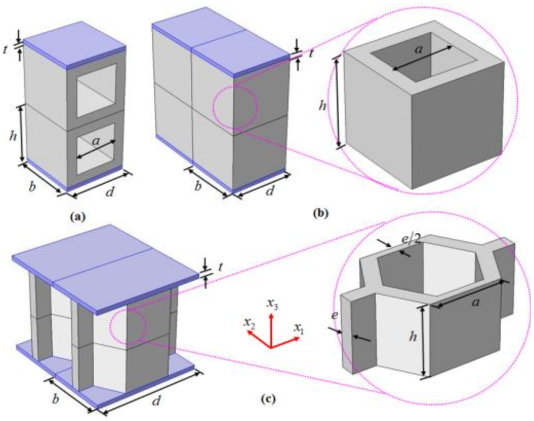

Fig. 2 Unit cells for sandwich beam with various cross-sections

- For the sandwich beam with square cores, the geometric variables are given by $b = d = h = 1.5[\text{m}]$, $t = 0.1[\text{m}]$, $\alpha = 1[m]$
- For the sandwich beam with 90-rotated square cores, the geometric variables are the same as above
- For the sandwich beam with hexagon cores, the geometric variables and given by $b = 1.3321[\text{m}]$, $d = 2.3072[\text{m}]$, $\alpha = 1[\text{m}]$, $e = 0.2[\text{m}]$

All sandwich beams in the above case have the same core material properties of $E_c = 3.5[\text{GPa}]$, $\nu_C = 0.34$ and face sheet material properties of $E_f = 70[\text{GPa}]$, $\nu_C = 0.34$. The effective bending and shear stiffness predicted by the analytical formulas in Dai and Zhang[8] and the present approach are listed in Table 1.

Table 1 Effective beam bending and shear stiffness ($\overline{X}_{33}$ and $\overline{G}_{11}$) of sandwich beams predicted by different methods ($\times 10^{10}[\text{N}\cdot\text{m}^2]$)

	Dai and Zhang [8]		The present approach	
	$\overline{X}_{33}$	$\overline{G}_{11}$ ($k = 1.2$)	$\overline{X}_{33}$	$\overline{G}_{11}$
Rectange cores	5.532	0.863	5.576	0.829
90 rectangle cores	10.221	1.358	11.022	1.357
Hexagon cores	4.139	1.182	4.181	1.195

As expected, our predictions show a good agreement with those in Dai and Zhang[8] with the biggest difference (around 8%) appearing for effective bending stiffness of the 90-rotated rectangle cores. In addition, to show the accuracy of the present approach, we analyzed a cantilevered beam composed of 10 UCs with 90-rotated hexagons using the full 3D FEA (COMSOL) (see Fig.1). With 10 UCs along the span, the 3D displacement distribution predicted by the Euler-Bernoulli beam model and the present model is plotted against the full 3D FEA in Fig. 3.

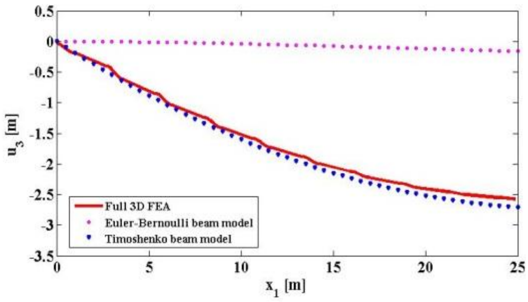

Fig. 3 3D displacement distribution along the beam axis at $x_2 = x_3 = 0.9$[m]

As expected, this result for the thin-walled hexagon sandwich beam illustrates the need for the Timoshenko beam model; to investigate the behaviors of such structures in which the effects of transverse shear deformations are significant because of less than 5 % off between the full 3D FEA and the Timoshenko beam model, and 96 % off between the full 3D FEA and the Euler-Bernoulli one.

5. Conclusions

In the present paper, VAM leading to simultaneous homogenization and dimensional reduction is used to construct a refined model for investigating the micro-mechanical behavior of beam-like structures. Also this model serves as a rigorous link between the original 3D heterogeneous problem of beam-like structures with complex micro-structures and the engineering beam model such as the Timoshenko beam model. Moreover, for the purpose of dealing with realistic and complex geometries and constituent materials at the microscopic level, the proposed approach is incorporated into a commercial analysis package, COMSOL. As a preliminary validation of the present approach, several examples available in literature are used to demonstrate both the consistency and efficiency of this new model, especially for the structures in which the effects of transverse shear deformation are significant. Clearly the present approach can accurately reproduce the full 3D FEA without significant loss of accuracy or significant computational costs.

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